Math 2130
HW 5 Solutions

① (a)

$$\int_1^3 \int_0^1 (1+4xy)\,dx\,dy$$

$$= \int_1^3 \Big[\underbrace{x + 4\cdot\frac{x^2}{2}y}_{x+2x^2y} \Big]_{x=0}^{1} dy$$

$$= \int_1^3 \Big[\big(1+2(1)^2 y\big) - \big(0 + 2\cdot 0^2 y\big) \Big] dy$$

$$= \int_1^3 (1+2y)\,dy = y + y^2 \Big|_1^3$$

$$= (3+3^2) - (1+1^2) = \boxed{10}$$

1 (b)

$$\int_0^2 \int_0^{\pi/2} x \sin(y)\, dy\, dx$$

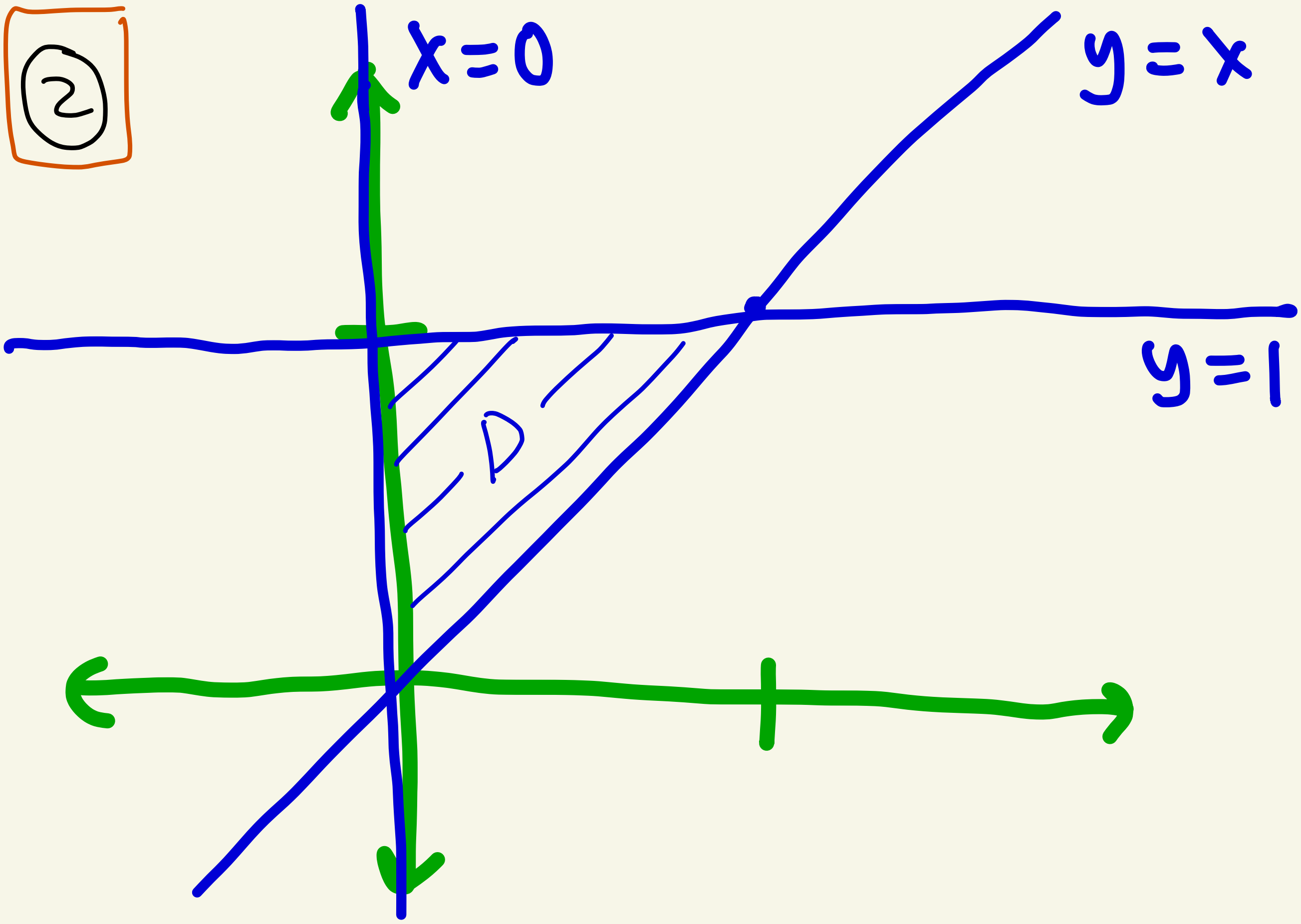
$$= \int_0^2 \Big[x\,(-\cos(y)) \Big]_{y=0}^{\pi/2} dx$$

$$= \int_0^2 \Big[-x \underbrace{\cos(\pi/2)}_{0} - \big(-x \underbrace{\cos(0)}_{1} \big) \Big] dx$$

$$= \int_0^{2\pi} x\, dx = \frac{1}{2} x^2 \Big|_0^{2\pi}$$

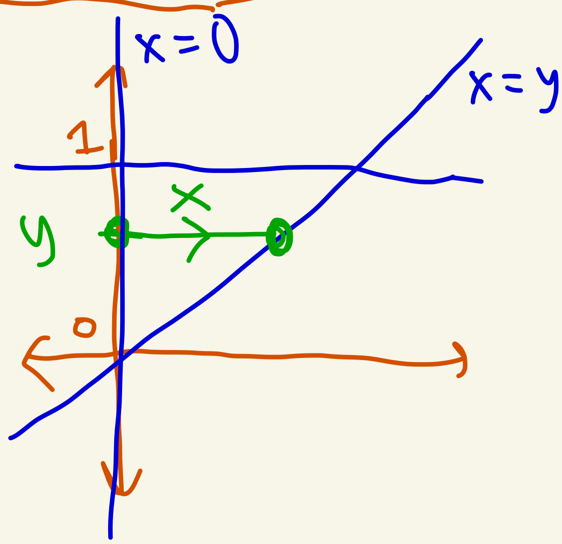
$$= \frac{1}{2}\left((2\pi)^2 - 0^2 \right) = \frac{4\pi^2}{2} = \boxed{2\pi^2}$$

2



Two ways to parameterize.

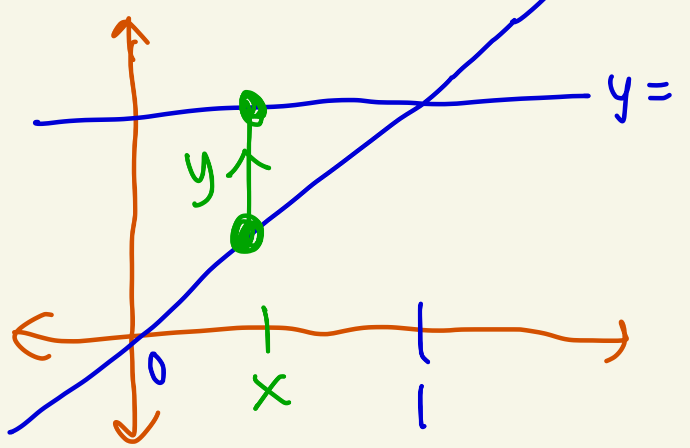
Way 1



$0 \le y \le 1$

$0 \le x \le y$

Way 2



$0 \le x \le 1$

$x \le y \le 1$

Way 1

$$\int_0^1 \int_0^y (x^2+3y^2)\,dx\,dy = \int_0^1 \left[\frac{1}{3}x^3+3y^2x\right]_{x=0}^{y} dy$$

$$= \int_0^1 \left[\left(\frac{1}{3}y^3+3y^2\cdot y\right)-\left(\frac{1}{3}\cdot 0^3+3y^2\cdot 0\right)\right] dy$$

$$= \int_0^1 \frac{10}{3}y^3\,dy = \frac{10}{3}\cdot\frac{1}{4}y^4\Big|_0^1 = \frac{5}{6}\left(1^4-0^4\right) = \boxed{\frac{5}{6}}$$

Way 2

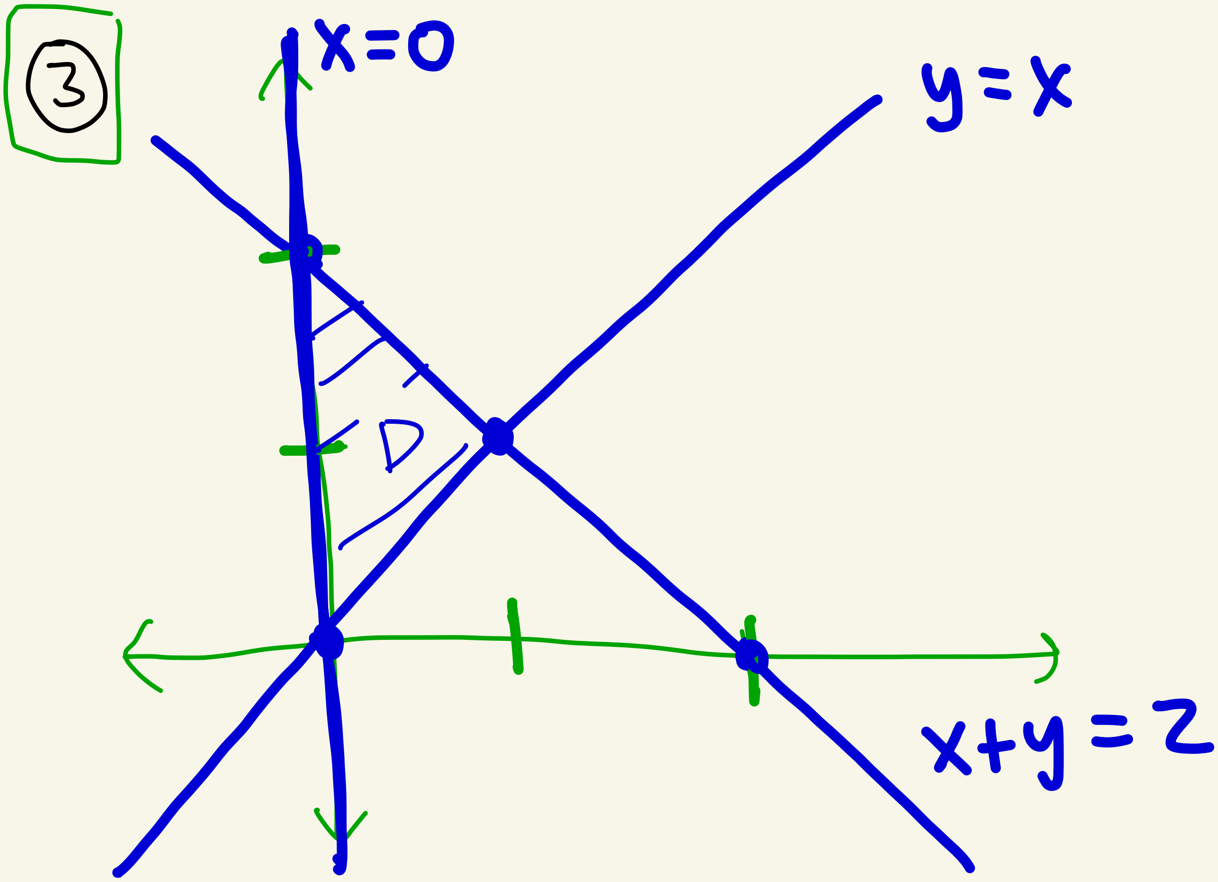
$$\int_0^1 \int_x^1 (x^2+3y^2)\,dy\,dx = \int_0^1 \left[x^2y+y^3\right]_{y=x}^{1} dx$$

$$= \int_0^1 \left[\left(x^2\cdot 1+1^3\right)-\left(x^2\cdot x+x^3\right)\right] dx$$

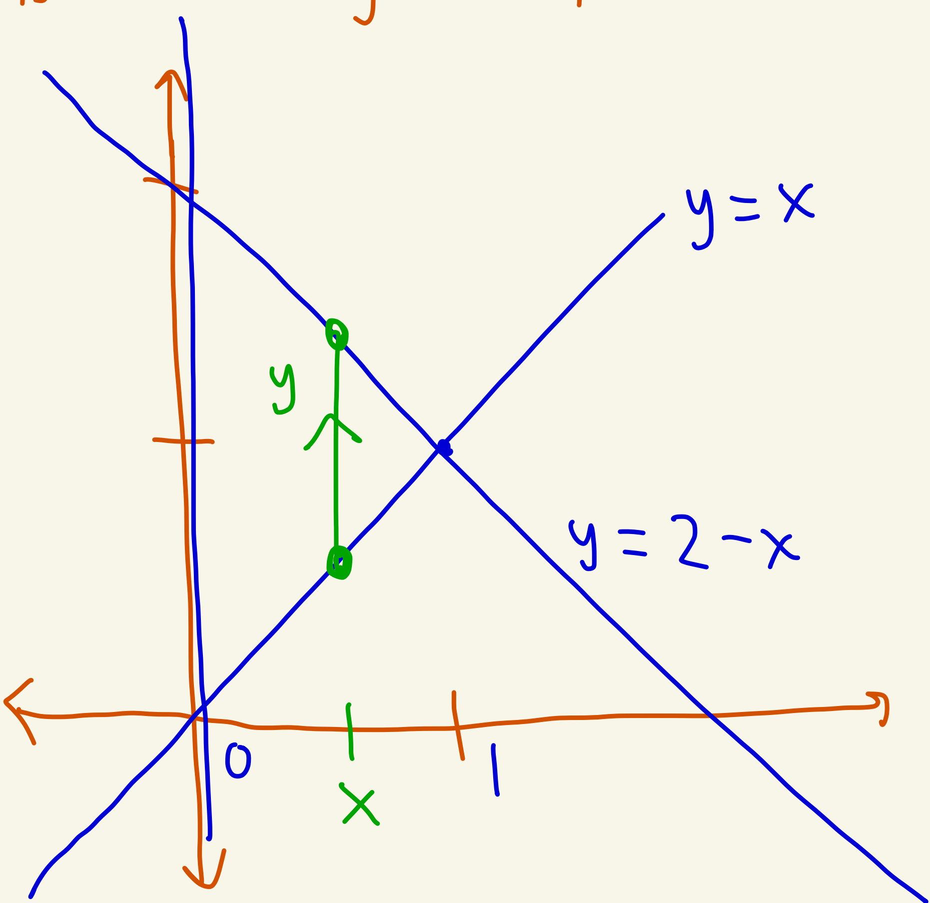
$$= \int_0^1 \left(-2x^3+x^2+1\right) dx = \left(-2\cdot\frac{1}{4}x^4+\frac{1}{3}x^3+x\right)\Big|_0^1$$

$$= \left(-\frac{1}{2}\cdot 1^4+\frac{1}{3}\cdot 1^3+1\right)-(0) = -\frac{1}{2}+\frac{1}{3}+1 = \boxed{\frac{5}{6}}$$

③



Best way to parameterize is this way:



$$\boxed{\begin{array}{l} 0 \le x \le 1 \\ x \le y \le 2-x \end{array}}$$

I show another way down below after this way

$$\int_0^1 \int_x^{2-x} x \, dy \, dx = \int_0^1 \left(xy \Big|_{y=x}^{2-x} \right) dx$$

$$= \int_0^1 \Big(x(2-x) - x(x) \Big) dx$$

$$= \int_0^1 (2x - x^2 - x^2) \, dx$$

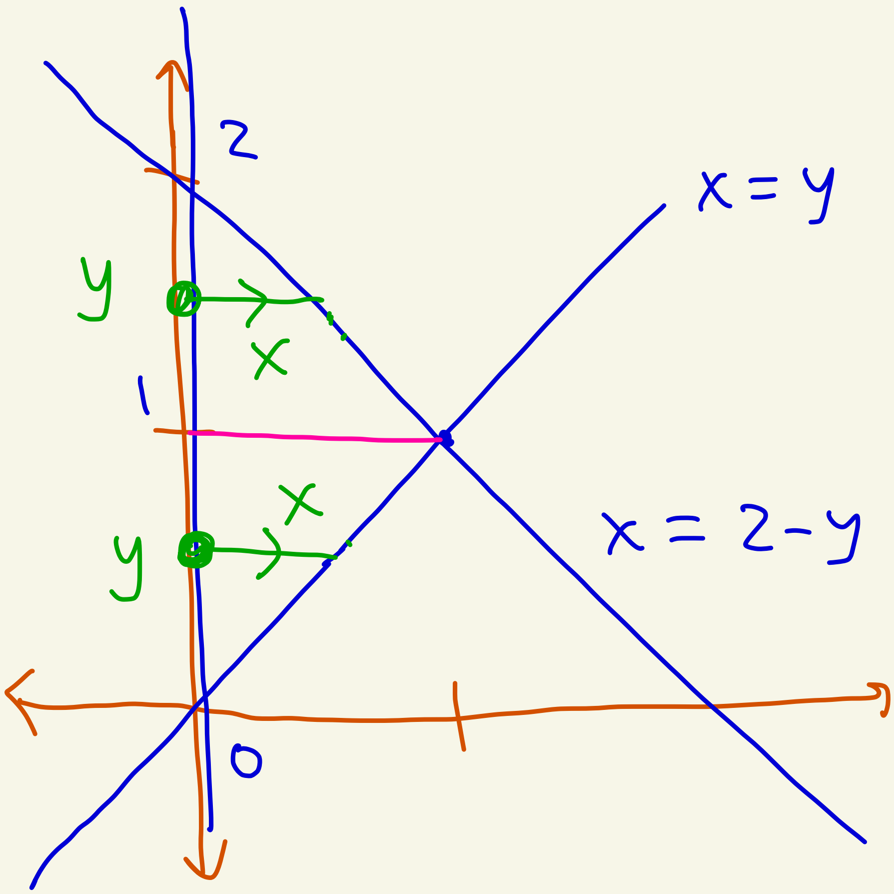
$$= \int_0^1 (-2x^2 + 2x) \, dx$$

$$= \left(-\frac{2}{3}x^3 + x^2 \right) \Big|_0^1$$

$$= \left(-\frac{2}{3}(1)^3 + (1)^2 \right) - \left(-\frac{2}{3}(0)^3 + (0)^2 \right)$$

$$= -\frac{2}{3} + 1 = \boxed{\frac{1}{3}}$$

You could do ③ above another way but you'd need two integrals.

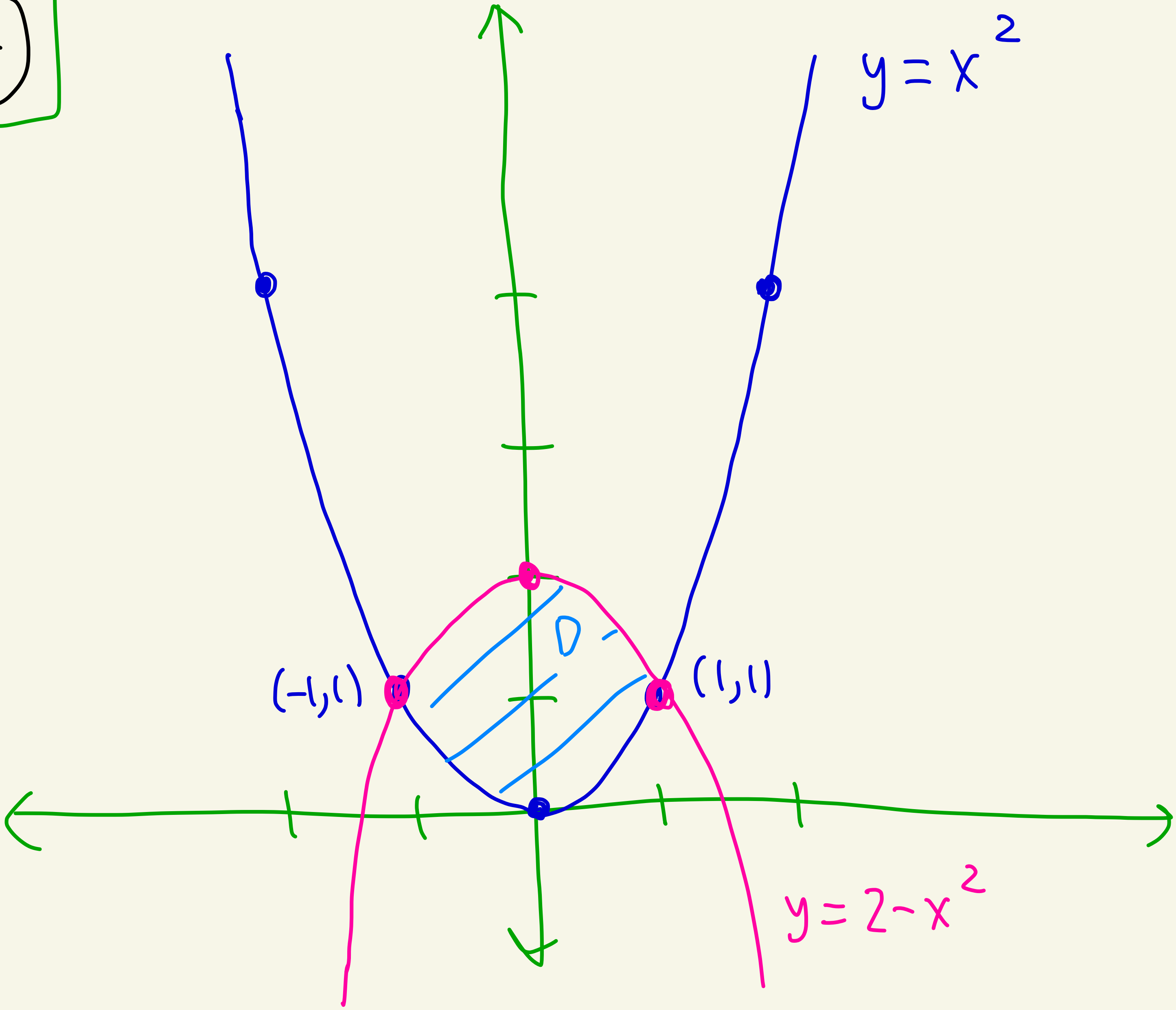


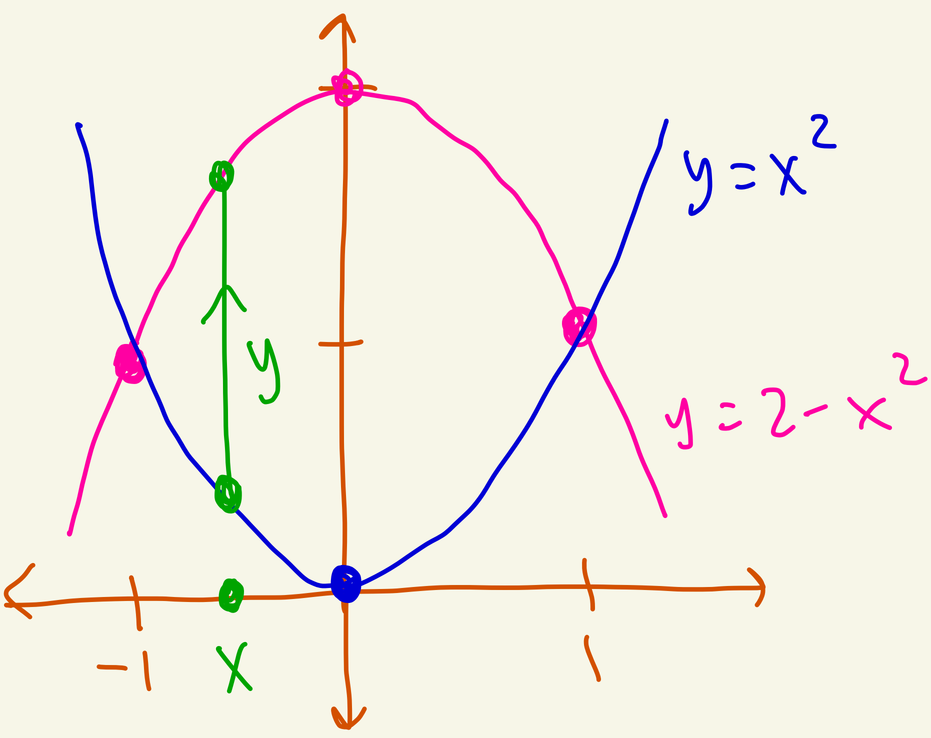
It would be

$$\int_0^1 \int_0^y x \, dx \, dy + \int_1^2 \int_0^{2-y} x \, dx \, dy$$

You can try it if you want to.

4





$$\boxed{\begin{array}{l} -1 \le x \le 1 \\ 2-x^2 \le y \le x^2 \end{array}}$$

$$\int_{-1}^{1}\int_{x^2}^{2-x^2} 5\,dy\,dx$$

$$= \int_{-1}^{1}\left[5y \Big|_{y=x^2}^{2-x^2} \right] dx$$

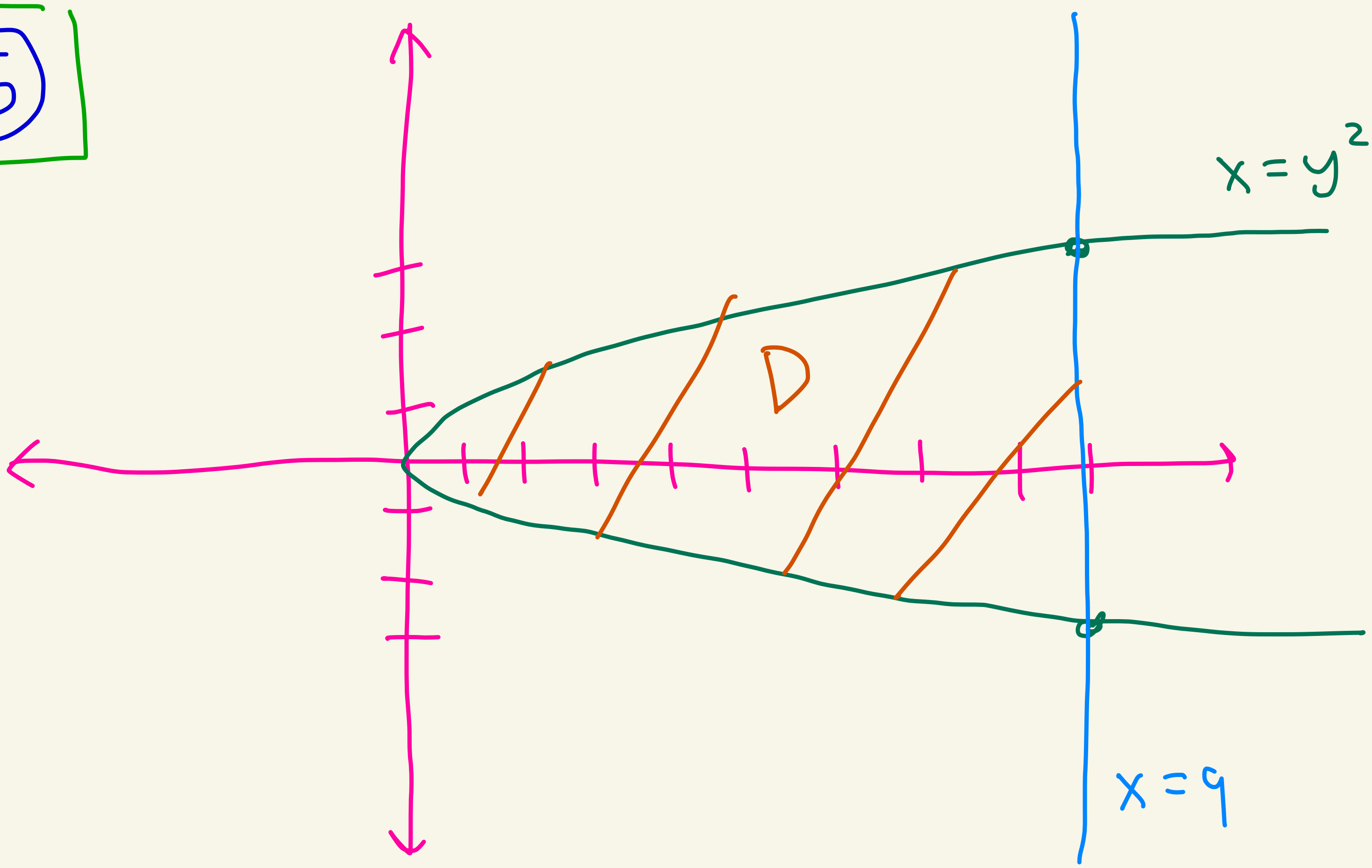
$$= \int_{-1}^{1}\left(5(2-x^2)-5x^2\right) dx$$

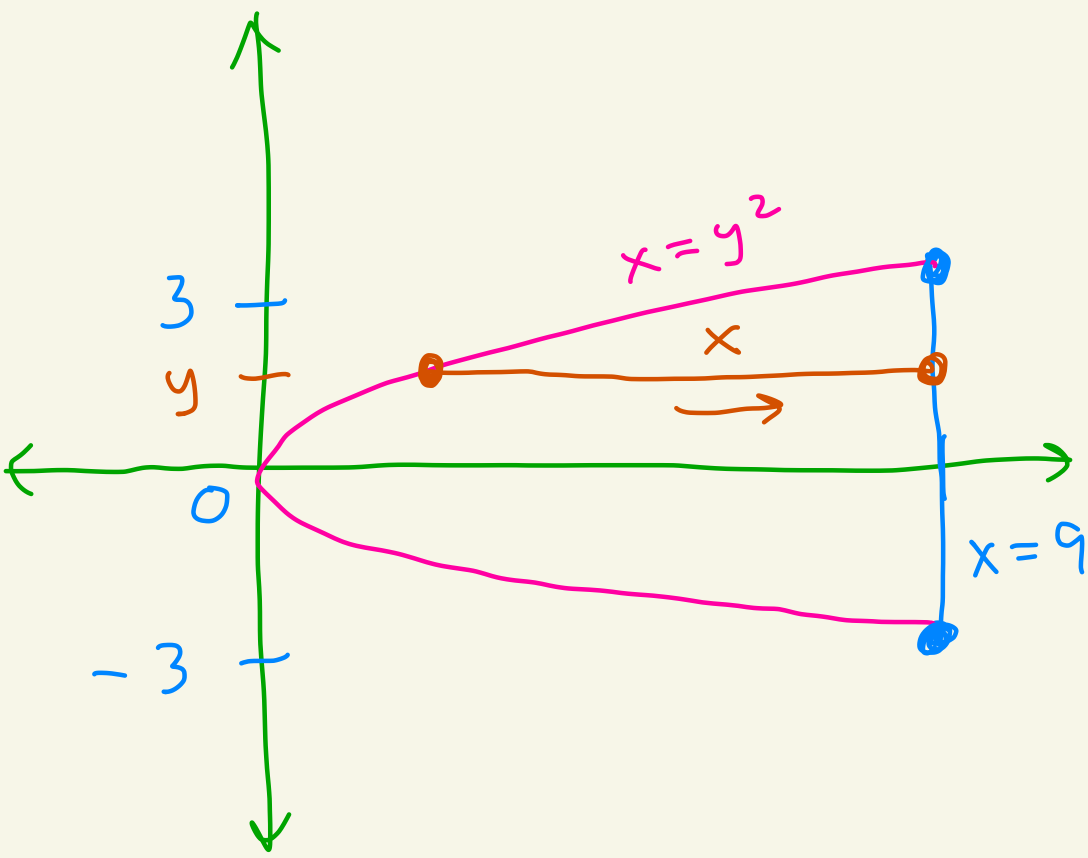
$$= \int_{-1}^{1}(10-10x^2)\,dx = \left(10x-\frac{10}{3}x^3\right)\Big|_{-1}^{1}$$

$$= \left(10(1)-\frac{10}{3}(1)^3\right)-\left(10(-1)-\frac{10}{3}(-1)^3\right)$$

$$= 10-\frac{10}{3}+10-\frac{10}{3} = 20-\frac{20}{3} = \frac{60-20}{3} = \boxed{\frac{40}{3}}$$

5





$$\boxed{\begin{array}{l} -3 \le y \le 3 \\ y^2 \le x \le 9 \end{array}}$$

$$\int_{-3}^{3}\int_{y^2}^{9}(x^2+y^2)\,dx\,dy = \int_{-3}^{3}\left(\frac{x^3}{3}+y^2x\right)\Bigg|_{x=y^2}^{9} dy$$

$$= \int_{-3}^{3}\left[\underbrace{\left(\frac{9^3}{3}+y^2\cdot 9\right)}_{243+9y^2} - \underbrace{\left(\frac{(y^2)^3}{3}+y^2\cdot y^2\right)}_{-\left(\frac{1}{3}y^6+y^4\right)}\right] dy$$

$$= \int_{-3}^{3}\left(-\frac{1}{3}y^6 - y^4 + 9y^2 + 243\right) dy$$

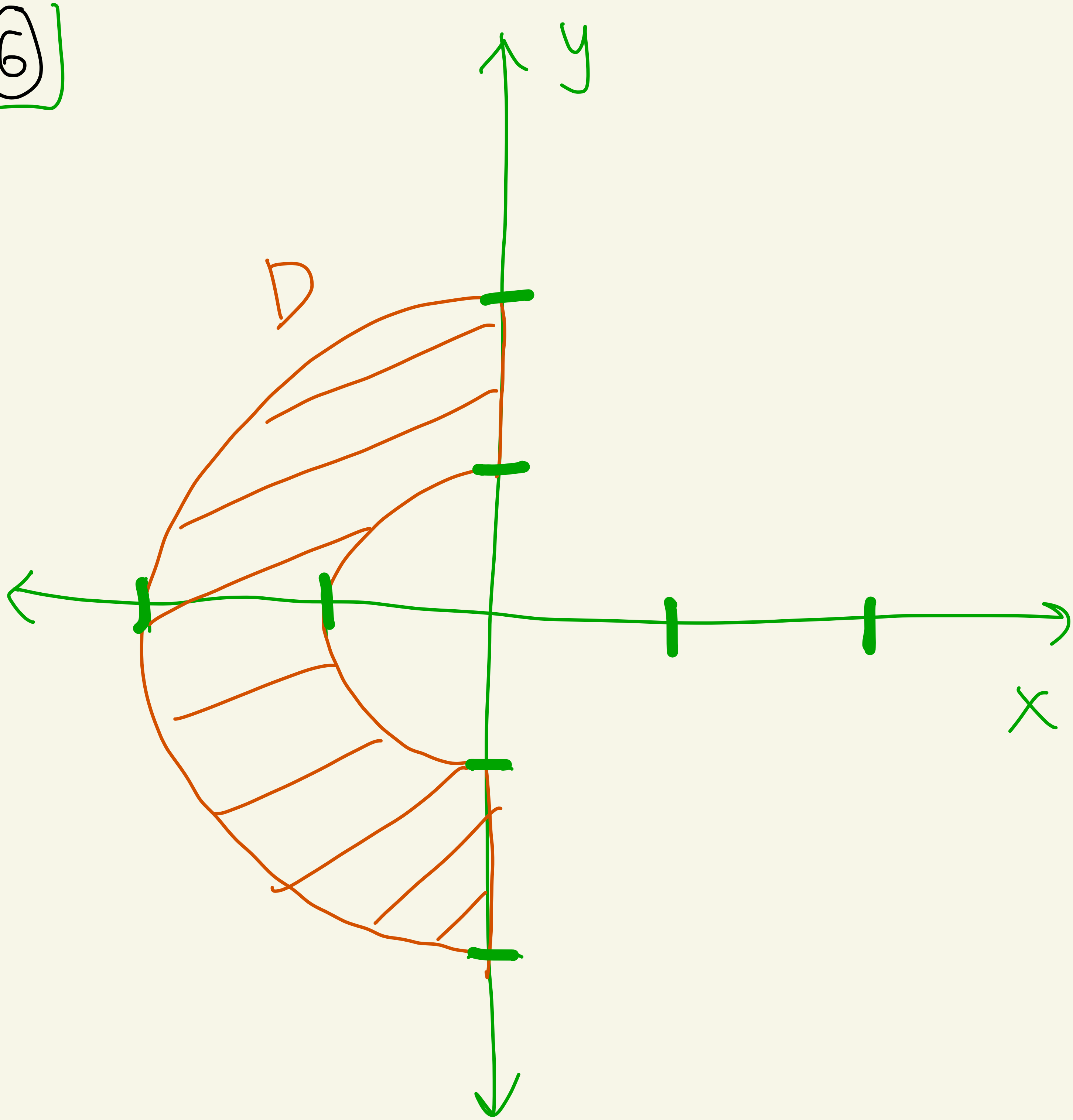
$$= \left[-\frac{1}{3}\,\frac{y^7}{7} - \frac{y^5}{5} + \frac{9}{3}y^3 + 243y\right]_{y=-3}^{3}$$

$$= \left[-\frac{1}{3}\cdot\frac{3^7}{7} - \frac{1}{5}\cdot\frac{3^5}{5} + 3\cdot(3)^3 + 243(3)\right]$$

$$- \left[-\frac{1}{3}\,\frac{(-3)^7}{7} - \frac{1}{5}\cdot\frac{(-3)^5}{5} + 3\cdot(-3)^3 + 243(-3)\right]$$

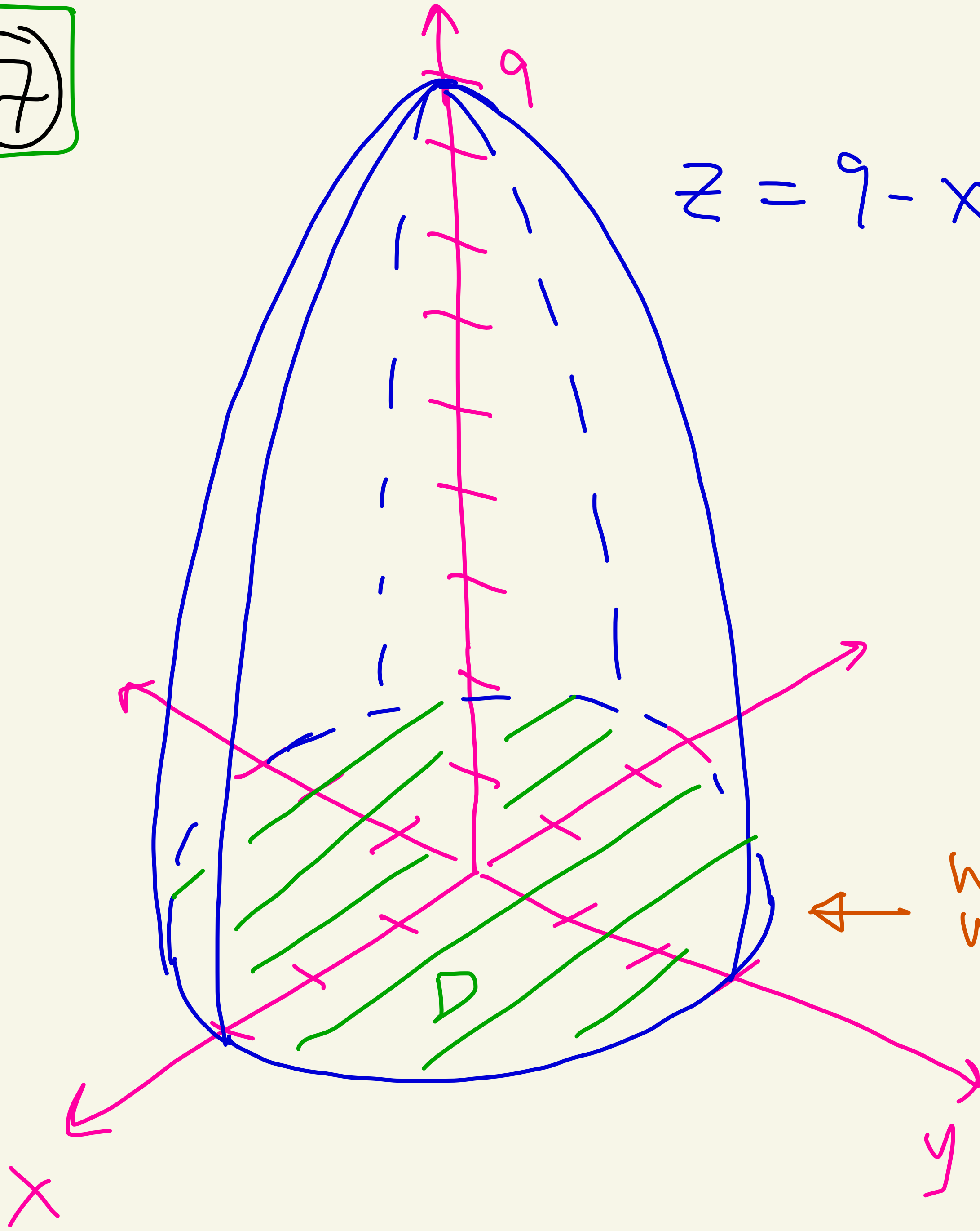
$$\approx \boxed{1392.27}$$

6

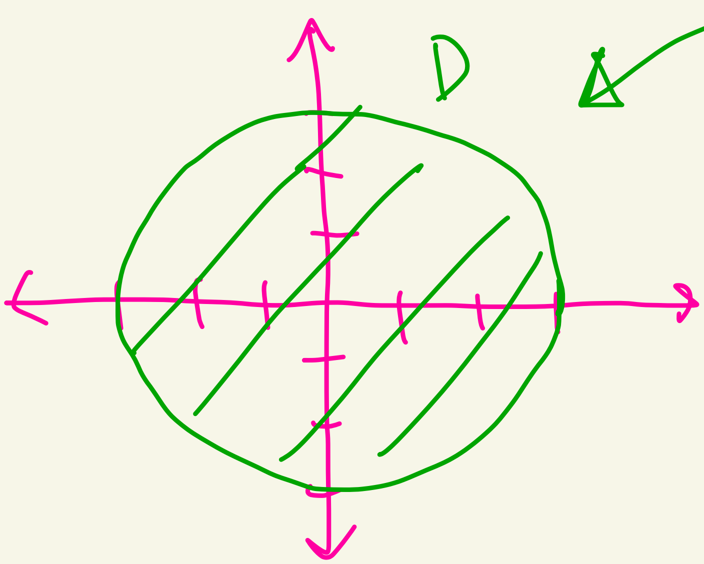


7

$z = 9 - x^2 - y^2$



When $z = 0$
we get
$0 = 9 - x^2 - y^2$
$x^2 + y^2 = 9.$



D

$0 \le r \le 3$
$0 \le \theta \le 2\pi$

$$\iint_D (9-x^2-y^2)\,dA$$

$dA = r\,dr\,d\theta$
$x^2+y^2=r^2$

$$= \int_0^{2\pi}\int_0^3 (9-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (9r-r^3)\,dr\,d\theta$$

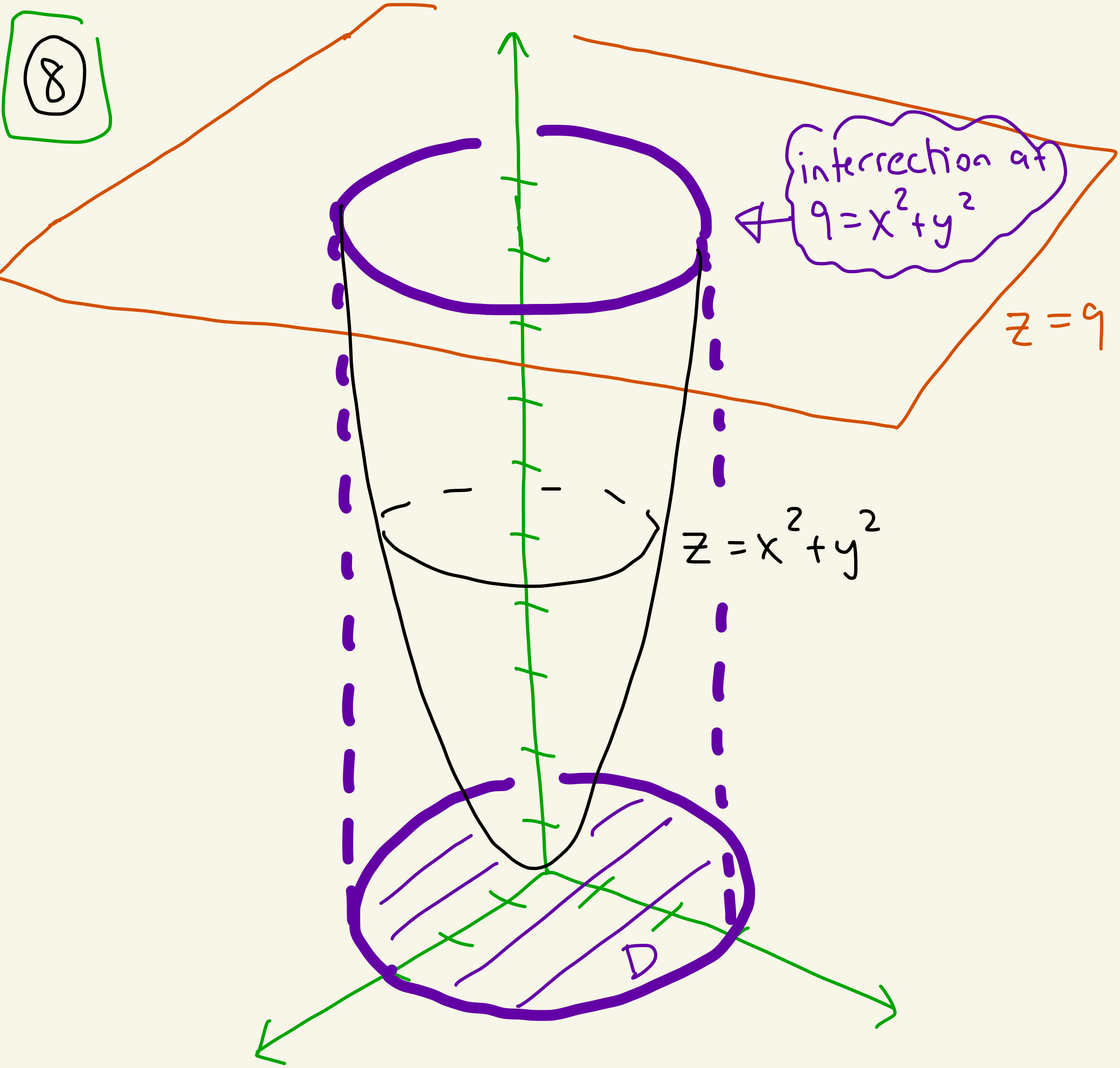
$$= \int_0^{2\pi} \left[9\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{3} d\theta$$

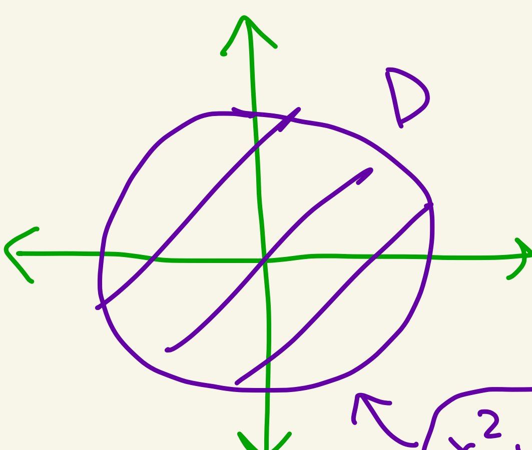
$$= \int_0^{2\pi} \left[\frac{9}{2}\cdot 3^2 - \frac{1}{4}\cdot 3^4\right] - [0-0]\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{81}{2} - \frac{81}{4}\right] d\theta = \int_0^{2\pi} \frac{81}{4}\,d\theta = \frac{81}{4}\theta\Big|_0^{2\pi}$$

$$= \frac{81}{4}\theta\Big|^{2\pi} = \qquad = \frac{81}{4}(2\pi) = \boxed{\frac{81}{2}\pi}$$

8





$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$$\iint_D \underbrace{\left[9-(x^2+y^2)\right]}_{(\text{top})-(\text{bottom})} dA$$

$dA = r\,dr\,d\theta$

$r^2 = x^2+y^2$

$$= \int_0^{2\pi}\int_0^{3} \left[9-r^2\right] r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3} \left(9r - r^3\right) dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{9}{2}r^2 - \frac{1}{4}r^4\right)\Big|_{r=0}^{3} d\theta$$

$$= \int_0^{2\pi} \left(\left[\frac{9}{2}\cdot 3^2 - \frac{1}{4}\cdot 3^4\right] - \left[0-0\right]\right) d\theta$$

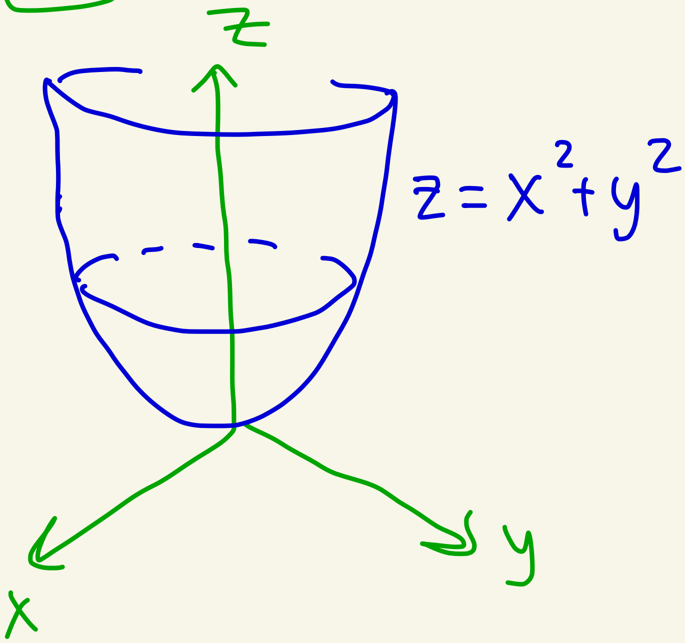
$$= \int_0^{2\pi} \left[\frac{81}{2} - \frac{81}{4} \right] d\theta$$

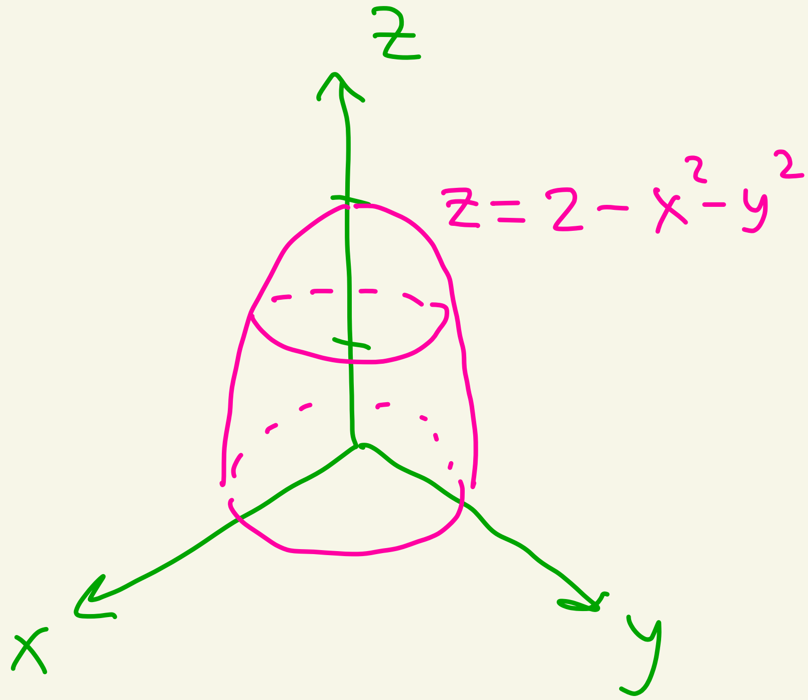
$$= \int_0^{2\pi} \frac{81}{4} \, d\theta = \frac{81}{4} \theta \Big|_0^{2\pi}$$

$$= \frac{81}{4} \left[2\pi - 0 \right]$$

$$= \boxed{\frac{81}{2} \pi}$$

9





Where do these surfaces intersect?

They intersect when:

$$x^2 + y^2 = 2 - x^2 - y^2$$

set z-values equal

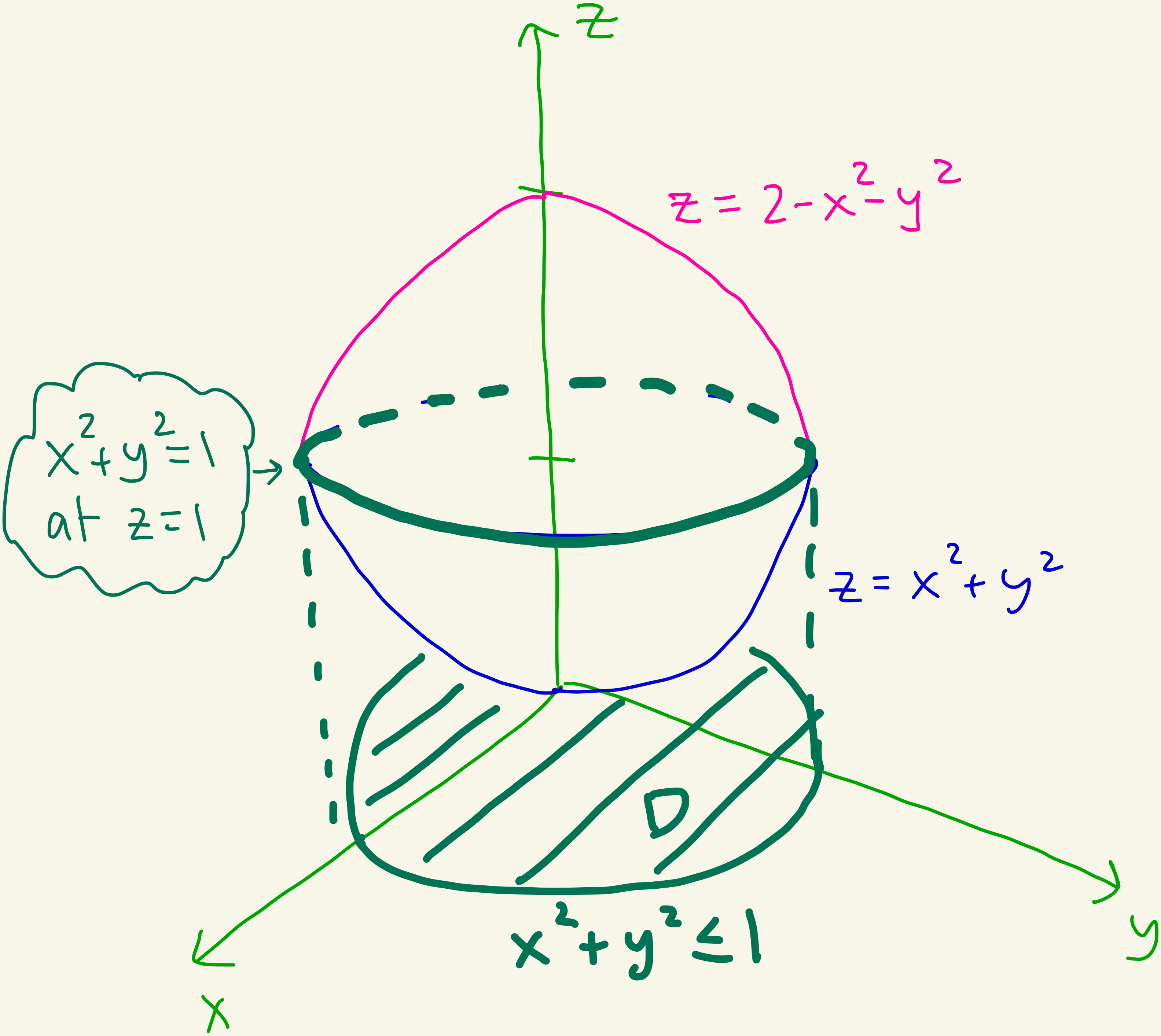
$$2x^2 + 2y^2 = 2$$

$$x^2 + y^2 = 1$$

This happens when $z = 1$

using the first surface $z = x^2 + y^2$

So we get the following picture:



The volume is

$$\iint_D \underbrace{(2 - x^2 - y^2)}_{\text{top}} - \underbrace{(x^2 + y^2)}_{\text{bottom}} \, dA$$

D is
$0 \le r \le 1$
$0 \le \theta \le 2\pi$

$dA = r\,dr\,d\theta$

$$= \int_0^{2\pi} \int_0^1 \left[(2 - r^2) - (r^2) \right] r \, dr \, d\theta$$

$r^2 = x^2 + y^2$

$$= \int_0^{2\pi} \int_0^1 (2r - 2r^3) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(r^2 - \frac{2}{4} r^4 \right) \Big|_{r=0}^{1} d\theta$$

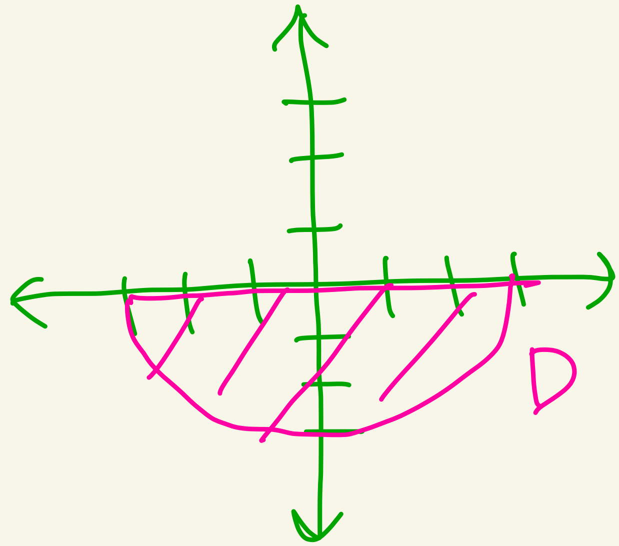
$$= \int_0^{2\pi} \left(1^2 - \frac{1}{2}(1)^4 \right) - \left(0^2 - \frac{1}{2}(0)^4 \right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \, d\theta = \frac{1}{2} \theta \Big|_0^{2\pi} = \frac{1}{2}(2\pi - 0)$$

$$= \boxed{\pi}$$

$\boxed{10}$

$$\iint_D e^{-x^2-y^2}\,dA$$



$0 \le r \le 3$

$\pi \le \theta \le 2\pi$

$r^2 = x^2 + y^2$

$dA = r\,dr\,d\theta$

$$= \int_{\pi}^{2\pi}\int_0^3 e^{-r^2}\, r\,dr\,d\theta$$

$$= \int_{\pi}^{2\pi}\left[\int_0^{-9}\left(-\frac{1}{2}e^{u}\right)du\right]d\theta$$

$u = -r^2$

$du = -2r\,dr$

$-\frac{1}{2}du = r\,dr$

$r = 0 \rightarrow u = -0^2 = 0$

$r = 3 \rightarrow u = -(3)^2 = -9$

$$= \int_{\pi}^{2\pi} \left(-\frac{1}{2}e^{u}\right)\Big|_{u=0}^{-9} d\theta$$

$$= \int_{\pi}^{2\pi} \left[-\frac{1}{2}e^{-9} - \left(-\frac{1}{2}e^{0}\right)\right] d\theta$$

$$= \int_{\pi}^{2\pi} \left[-\frac{1}{2}e^{-9} + \frac{1}{2}\right] d\theta$$

$$= \left[-\frac{1}{2}e^{-9} + \frac{1}{2}\right] \theta \Big|_{\theta=\pi}^{2\pi}$$

$$= \left[-\frac{1}{2}e^{-9} + \frac{1}{2}\right](2\pi - \pi)$$

$$= \boxed{\frac{\pi}{2}\left(1 - e^{-9}\right)} = \boxed{\frac{\pi}{2}\left(1 - \frac{1}{e^{9}}\right)}$$